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A CLASS OF INTEGRAL EQUATIONS OF THE AXISYMMETRIC THEORY OF ELASTICITY FOR A COMPOSITE SPACE WITH GAPS AT THE INTERFACE†

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A space made up of two different kinds of elastic half-spaces with a coaxial system of circular and annual gaps at the interface is considered. Loads, which are solely dependent on a radial coordinate, are specified at the edges of the gaps. A representation of its general solution in terms of generalized analytic functions is used to solve this problem in the theory of elasticity. A singular integral equation is obtained after satisfying the boundary conditions at the gaps and carrying out a number of calculations. This equation is subsequently reduced to a Fredholm equation by an exact inversion of the characteristic part. A closed solution is obtained in the case of a single circular gap.

The reduction of problems in the plane theory of elasticity for bodies with cracks to a problem of the matching of complex analytic functions on the crack boundaries and then reducing them to integral equations is an effective method for solving them. In spatial, axisymmetric problems in the theory of elasticity in the case of a plane boundary it is also possible in a number of cases to reduce them to matching problems and, then, to integral equations using p-analytic functions. Some mixed problems in the theory of elasticity have been solved using this method in the axisymmetric case for a half-space, a space with coaxial circular gaps, etc. [1–4]. Some similar problems were solved in [1] by reducing them to a matching problem for analytic functions. The use of generalized analytic functions in this kind of problem is also effective and, in particular, in the case of an elastic space composed of two different half-spaces when there are annular slots in the plane of separation, with which this paper is concerned.

Let the elastic characteristics of the material in the case of each of the two half-spaces be different and equal respectively to G_1, ν_1 and G_2, ν_2, where G_j is the shear modulus and ν_j is Poisson's ratio ($j = 1, 2$). Assuming that there is ideal contact everywhere along the plane of the interface of the half-spaces $z < 0$, $0 < r < \infty$ and $z > 0$, $0 < r < \infty$ (z and r are cylindrical coordinates), apart from the domains which correspond to the gaps, and that the loads on the boundaries of the gaps are specified, we can write the boundary conditions on the boundaries of the gaps, using generalized analytic functions, in the form [1]

$$\Phi_k(t) - \overline{\Psi_k(t)} = f_k(t) = \sigma_z^{(k)} + i\tau_{zr}^{(k)} \tag{1}$$

$$\left(L = \sum_{k=0}^{n} L_k;\;\; L_k \to (z = 0, a_k < |r| < b_k); k = 0, 1, \ldots, n \right)$$

The matching conditions on the common boundary of the half-spaces, that is, equality of the loads and, also, the displacements $u_z^{(k)}$ and $u_r^{(k)}$ are

$$\Phi_1(t) - \overline{\Psi_1(t)} = \Phi_2(t) - \overline{\Psi_2(t)} \tag{2}$$

$$(\varkappa_1\varphi_1(t) - \overline{\psi_1(t)}) / G_1 = (\varkappa_2\varphi_2(t) - \overline{\psi_2(t)}) / G_2 \quad (z \notin L) \tag{3}$$

where $\Phi_k(t)$, $\Psi_k(t)$ are generalized analytic functions when $z < 0$ ($k = 1$) and $z > 0$ ($k = 2$), respectively, $\sigma_z^{(k)}$ and $\sigma_r^{(k)}$ are the normal and shear stresses specified on the boundaries of the gaps, $u_z^{(k)}$ and $u_r^{(k)}$ are the displacements and $\Phi_k(t) = \varphi'(t)$, $\Psi_k(t) = \psi'_k(t)$, $\varkappa_k = 3 - 4\nu_k$, $a_{n+1} = \infty$.

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In the case of generalized analytic functions, the operation of differentiation is defined as follows [1]:

$$\varphi'(t) = \lim_{t_1 \to t} (\varphi(t_1) - \operatorname{Re}\varphi(t) - irr_1^{-1} \operatorname{Im}\varphi(t))(z - z_1 + i(r - r_1))^{-1} \tag{4}$$

$$(t = z + ir, \quad t_1 = z_1 + ir_1)$$

The stresses and strains are determined in terms of generalized analytic functions using the formulae

$$\begin{aligned} &\sigma_z + \sigma_r + \sigma_\theta = 4(1+\nu)\operatorname{Re}\Phi(t) \\ &\sigma_\theta = 4\nu \operatorname{Re}\Phi(t) + 2Gu_r / r \\ &\sigma_z + i\tau_{zr} = \Phi(t) - 2z\overline{\Phi'(t)} - \overline{\Psi(t)} \\ &2G(u_z + iu_r) = \varkappa\varphi(t) - 2z\overline{\varphi'(t)} - \overline{\psi(t)} \end{aligned} \tag{5}$$

Assuming the derivatives of $\varphi_k(t)$ and $\psi_k(t)$ exist at the interface, differentiating (3) using formula (4), and allowing for the fact that $(\overline{\varphi(t)})' = -\overline{\varphi'(t)}$ when $t = ir$, instead of (3), we obtain

$$(\varkappa_1\Phi_1(t) + \overline{\Psi_1(t)}) / G_1 = (\varkappa_2\Phi_2(t) + \overline{\Psi_2(t)}) / G_2 \tag{6}$$

We transform (2) and (6) to the form

$$\begin{aligned} &(1 + \lambda\varkappa_1)\Phi_1(t) - (1 - \lambda)\overline{\Psi_1(t)} = (1 + \varkappa_2)\Phi_2(t) \\ &(\varkappa_2 - \lambda_1\varkappa_1)\Phi_1(t) - (\varkappa_2 + \lambda)\overline{\Psi_1(t)} = (1 + \varkappa_2)\Psi_2(t) \end{aligned}$$

Passing in the second of these to the conjugate values and then expressing them in terms of the remaining terms, we obtain

$$\begin{aligned} &(1 + \lambda\varkappa_1)\Phi_1(t) = (1 - \lambda)\overline{\Psi_1(t)} + (1 + \varkappa_2)\Phi_2(t) \\ &(\varkappa_2 + \lambda)\Psi_1(t) = (\varkappa_2 - \lambda\varkappa_1)\overline{\Phi_1(t)} + (1 + \varkappa_2)\Psi_2(t) \end{aligned} \tag{7}$$

It follows from (7) that $\Phi_1(t)$ and $\Psi_1(t)$ are analytically continuable into the domain $z > 0$ since the functions $\Phi_1(-t)$ and $\Phi_1(-t)$ are generalized analytic functions when $z > 0$ as $\Phi_1(t)$ and $\Psi_1(t)$ are generalized analytic functions when $z < 0$.

From (7) we obtain

$$\begin{aligned} &\Phi_2(t) = \frac{1 + \lambda\varkappa_1}{1 + \varkappa_2}\Phi_1(t) - \frac{1 - \lambda}{1 + \varkappa_2}\Psi_1(-t) \\ &\Psi_2(t) = \frac{\varkappa_2 + \lambda}{\varkappa_2 + 1}\Psi_1(t) + \frac{\lambda\varkappa_1 - \varkappa_2}{\varkappa_2 + 1}\Phi_1(-t) \end{aligned} \tag{8}$$

Taking account of the fact that $\overline{\Phi(t)} = \Phi(\bar{t})$ when $t = ir$, we can write condition (1) in the form

$$\Phi_k(t) - \overline{\Psi_k(t)} = \Phi_k(t) - \Psi_k(-t) = \sigma_z^{(k)} + i\tau_{zr}^{(k)} \quad (k = 1, 2) \tag{9}$$

Since $t = z + ir \to -0 + ir$ when $z \to -0$ and the variable $\zeta = -t \to +0 - ir$, the function $\Psi_k(-t)$ in (8) is the boundary value of $\Psi_k(\zeta)$ on the boundary of the gap L_+ when $k = 1$ and on the boundary of the gap L_- when $k = 2$ ($\zeta = -t$, $t = \lim (z + ir) = \pm 0 + ir$ when $z \to \pm 0$). On replacing t by $-t(t = ir)$ in (9) and adding the right-hand and left-hand sides of (9) and of the equality obtained by this replacement, we obtain

$$\Phi_{1*}^-(t) - \Psi_{1*}^+(t) = f_{1*}(t), \quad \Phi_{2*}^+(t) - \Psi_{2*}^-(t) = f_{2*}(t) \quad (t \in L) \tag{10}$$

where

$$\Phi_{k*}(t)=\Phi_k(t)+\Phi_k(-t),\quad \Psi_{k*}(t)=\Psi_k(t)+\Psi_k(-t)$$
$$f_{k*}(t)=f_k(t)+f_k(-t)\quad (k=1,2)$$

Now, using (8) and carrying out similar operation with them and then substituting the resulting expressions for $\Phi_{2*}(t)$ and $\Psi_{2*}(t)$ into condition (10), we obtain

$$\Phi_*^-(t)-\Psi_*^+(t)=f_{1*}(t) \tag{11}$$
$$\Phi_*^+(t)-(1+\beta)\Psi_*^-(t)+\alpha\Phi_*^-(t)-\delta\Psi_*^+(t)=f_{2*}(t)\quad (t\in L)$$
$$\Phi_*(t)=\Phi_{1*}(t),\quad \Psi_*(t)=\Psi_{1*}(t)$$
$$\alpha=\frac{\varkappa_2-\lambda\varkappa_1}{1+\lambda\varkappa_1},\quad \delta=\frac{1-\lambda}{1+\lambda\varkappa_1};\quad \beta=\alpha-\delta$$

On multiplying the right- and left-hand sides of the first of equalities (11) by $1+\alpha$ and then subtracting the right- and left-hand sides of the second quality of (11) from the right- and left-hand sides of the resulting equality, we obtain

$$[\Phi_*(t)+(1+\beta)\Psi_*(t)]^+-[\Phi_*(t)+(1+\beta)\Psi_*(t)]^-=$$
$$=f_{2*}(t)-(1+\alpha)f_{1*}(t)=F(t) \tag{12}$$

Hence, we obtain a matching problem for the function $\Omega(t)=\Phi_*(t)+(1+\beta)\Psi_*(t)$ which has the simple solution [5]

$$\Phi_*(\zeta)+(1+\beta)\Psi_*(\zeta)=F_1(\zeta)+A \tag{13}$$
$$F_1(\zeta)=\frac{1}{2\pi i}\int_L\frac{F(\tau)\omega(\zeta,\tau)}{\zeta-\tau}d\tau\quad (\zeta\in L)$$

where $A=0$ subject to the condition that $\Phi_*(\zeta)=\Psi_*(\zeta)=0$ when $|\zeta|\to\infty$. The latter holds subject to the condition that the stresses at infinity are equal to zero [1]. $W(\zeta,\tau)$ is a generalized Cauchy kernel [1] which can be written in the form

$$W(\zeta,\tau)=\omega(\zeta,\tau)/(\zeta-\tau)$$
$$\omega(\zeta,\tau)=\begin{cases}\left|\dfrac{\tau-\bar\tau}{\tau-\bar\zeta}\right|B(k) & (\operatorname{sgn}(\operatorname{Im}\zeta\operatorname{Im}\tau)>0)\\ \left|\dfrac{\tau-\bar\tau}{\tau-\zeta}\right|D(k_0) & (\operatorname{sgn}(\operatorname{Im}\zeta\operatorname{Im}\tau)<0)\end{cases}$$
$$B(k)=K(k)-D(k),\quad D(k)=(K(k)-E(k))/k^2$$
$$k=\sqrt{|\zeta-\bar\zeta||\tau-\bar\tau|}/|\tau-\bar\zeta|,\quad k_0=\sqrt{|\zeta-\bar\zeta||\tau-\bar\tau|}/|\tau-\zeta|$$

Along the line $z=0$, we have

$$k=2\sqrt{rx}/(r+x),\quad k_0=2\sqrt{rx}/(r-x)\quad (t=z+ir,\quad \tau=y+ix)$$

On substituting $\Psi_*(\zeta)$ from (13) into the first equality (11), we obtain

$$\gamma\Phi_*^+(t)+\Phi_*^-(t)=F_*(t);\quad \gamma=1/(1+\beta),\quad F_*(t)=f_{1*}(t)+\gamma F_1^+(t) \tag{14}$$

where $F^+(t)$ is the limiting value of $F(t)$ on L_+, the upper boundary of the gaps.

We know [1] that a function which is a generalized analytic function everywhere in a plane, apart from the line L, can be represented in the form of a Cauchy type integral

$$\Phi(\zeta)=\frac{1}{2\pi i}\int_L \varphi(\tau)\frac{\omega(\zeta,\tau)}{\zeta-\tau}d\tau \tag{15}$$

and, from formula (15) using the Sokhotskii–Plemelj formula for the boundary value of $\Phi^-(t)$ and $\Phi^+(t)$, we have

$$\Phi^{\pm}(\zeta)=\pm\varphi(\zeta)/2+\Phi(\zeta) \tag{16}$$

Substituting the expressions for $\Phi_*^+(t)$ and $\Phi_*^-(t)$ into condition (14), we obtain a singular equation for the function $\varphi_*(t)$

$$\left(\frac{\gamma-1}{2}\right)\varphi_*(t)+\left(\frac{\gamma+1}{2}\right)\frac{1}{\pi i}\int_L\frac{\varphi_*(\tau)d\tau}{t-\tau}+\int_L K(t,\tau)\varphi_*(\tau)d\tau=F_*(t) \tag{17}$$

Here

$$K(t,\tau)=k_*(t,\tau)/|\tau-\bar{t}|^{\alpha} \tag{18}$$

The function $k_*(t,\tau)$ satisfies the Hölder condition $H(\alpha)$ ($\alpha<1$) with respect to both variables [1].

According to the general theory of singular integral equations [2], the characteristic equation for Eq. (17), in the case being considered here, has the following solution which vanishes at infinity

$$X(t)=\prod_{k=1(0)}^{2n}((t-ia_k)(t-ib_k))^{-1/2}\left(\frac{t-ia_k}{t-ib_k}\right)^{\varepsilon i}(t^2+b_0^2)^{-1/2} \tag{19}$$

when there is a circular gap ($a_0=0$). When there is no circular gap, the last factor in (19) changes to unity. The index of the characteristic equation p is equal to $2n+1$ when there is a gap and $2n+2$ when there is no gap and $\varepsilon=\ln \nu/(p2\pi)$.

Expressions of the type (19) should be understood as branches which are holomorphic in the plane $t=z+ir$ cut along the arc L and fixed, for example, in L.

We will denote the operator [1]

$$K^*F=A^*(t_0)F(t_0)-B^*(t_0)\frac{Z(t_0)}{\pi i}\int_L\frac{F(t)dt}{Z(t)(t-t_0)}$$

$$A^*(t_0)=\frac{1-\gamma}{2\gamma},\quad B^*(t_0)=-\frac{1+\gamma}{2\gamma}$$

$$Z(t_0)=\gamma X^+(t_0),\quad X^+(t_0)=\left(-\frac{1}{\gamma}\right)^{1/2}X(t_0)$$

by K^*.

According to the theory of singular integral equations [6], Eq. (17) (subject to condition (18)) is equivalent to the Fredholm equation

$$\varphi_*(t_0)+K^*K\varphi_*(t_0)=f^*(t_0)\quad(t_0\in L)$$

$$f^*(t_0)=K^*F_*(t_0)+Q_{p-1}(t_0)B^*(t_0)Z(t_0) \tag{20}$$

where $Q_{p-1}(t_0)$ is an arbitrary polynomial of degree not higher than $(p-1)$. The conditions of the single-valuedness of the displacements have to be used in order to find the unknown coefficients of the polynomial $Q_{p-1}(t)$. These conditions consist of the fact that the expression $Y(t)$ which is defined by the formula

$$Y(t) = \varkappa_k \Phi_{k*}(t) - t\overline{\Phi'_{k*}(t)} - \overline{\Psi_{k*}(t)}/(2G_k)$$

$$k = \begin{cases} 1 \text{ when } & z < 0 \\ 2 \text{ when } & z > 0 \end{cases}$$

must return to its initial value when the point t describes closed contours Λ_k which encompass the segments $L_k = a_k b_k$.

On contracting the contours Λ_k to the segments, we see that the conditions of single-valuedness reduce to equalities

$$\int_{L_k} (\varkappa_1 \Phi^-_{1*}(\tau) + \Psi^+_{1*}(\tau)) G_1^{-1} d\tau + \int_{L_k} (\varkappa_2 \Phi^+_{2*}(\tau) + \Psi^-_{2*}(\tau)) G_2^{-1} d\tau = 0 \tag{21}$$

$$(k = 0, 1, 2, 3, \ldots, n)$$

where $\Phi^+_{2*}(\tau)$ and $\Psi^-_{2*}(\tau)$ are determined using formulae which follow from formulae (7)

$$\begin{aligned} \Phi_{2*}(\tau) &= \frac{1+\lambda\varkappa_1}{1+\varkappa_2} \Phi^+_{1*}(\tau) - \frac{1-\lambda}{1+\varkappa_2} \Psi^-_{1*}(\tau) \\ \Psi_{2*}(\tau) &= \frac{\varkappa_2+\lambda}{1+\varkappa_2} \Psi^+_{1*}(\tau) + \frac{\lambda\varkappa_1 - \varkappa_2}{1+\varkappa_2} \Phi^-_{1*}(\tau) \end{aligned} \tag{22}$$

In formulae (22), it is sufficient to take account of the contours L_k lying in the right-hand half of the complex plane $r > 0$ since the coefficients of the polynomial $Q_{p-1}(\tau)$ are real.

When there is a circular gap, it is also necessary to take account of its contour L_0.

When formulae (11) and the Sokhotskii–Plemelj formulae for the functions $\Phi_{1*}(\tau)$ and $\Psi_{1*}(\tau)$ are taken into consideration and the expressions for $\Phi^+{}_{2*}(\tau)$ and $\Psi^-{}_{2*}(\tau)$ from (22) are substituted into condition (21), we obtain a system of linear algebraic equations for finding the coefficients of the polynomial.

This system is always solvable.

In fact, the homogeneous system which is obtained in the case when $f^*(t_0) = 0$ does not have a solution apart from $C_0 = C_1 = \ldots = C_{2n} = 0$ ($C_0 = C_1 = \ldots = C_{2n+1} = 0$) or the initial problem has only the trivial solution $\Phi_*(\tau) = \Psi_*(\tau) = 0$ and this means, as follows from the Sokhotskii–Plemelj formulae (16), $\Phi_*(\tau) = 0$ that the assertion is proved (C_k are the coefficients of the polynomial $Q_{p-1}(\tau)$ ($k = 0, 1, \ldots, 2n$ ($2n + 1$))).

The antisymmetric problem for the functions $\Phi_{k**}(\tau) = \Phi_k(\tau) - \Phi_k(-\tau)$, $\Psi_{k**}(\tau) = \Psi_k(\tau) - \Psi_k(-\tau)$ is solved in exactly the same way. In order to determine the displacements and stresses, it is necessary still to find $\varphi_k(t)$ and $\psi_k(t)$ by integration.

Since it is assumed that the displacements vanish at infinity, it is possible to take $t_0 = -\infty + ir_0$ when $k = 1$ and $t_0 = +\infty + ir_0$ when $k = 2$. Then, from the definition of integration of generalized analytic functions [1], we obtain

$$\varphi_k(t) = \int_{t_0}^{t=z+ir_0} \Phi_k(\tau) dz, \quad \psi_k(t) = \int_{t_0}^{t=z+ir_0} \Psi_k(\tau) dz, \quad k = \begin{cases} 1 \text{ when } z < 0 \\ 2 \text{ when } z > 0 \end{cases}$$

It is seen that the conditions for the displacements to vanish at infinity are satisfied. After this, the displacements u_z and u_r are calculated using the last of formulae (5) and the stresses are found using the first three equations of (5).

When there is only a single circular gap, the solution of the problem can be obtained in closed form immediately from the matching problem (12). In the plane with an aperture, which intersects the z axis and is symmetric with respect to it (in the case under consideration, L_+ and L_-, the upper and lower edges of the gap respectively, serve as the boundary of the opening), the generalized analytic function $\Phi_*(t)$ outside the opening and which vanishes at infinity can be represented in the form [1]

$$\Phi_*(t) = S(\varphi_0(\zeta)) = -\frac{2}{\pi|t-\bar{t}|}\int_{\bar{t}}^{t}\varphi_0(\zeta)\sqrt{\frac{\zeta-\bar{t}}{\zeta-t}}d\zeta \tag{23}$$

where $\varphi_0(\zeta)$ is a function which is holomorphic everywhere in the plane, apart from L, and we shall also assume that the line of branching of the radical coincides with the lower boundary and that the line of integration passes below L_- or above L_+. The magnitude of the integral in (23) is then independent of the path of integration, subject to the condition that $\lim \zeta\varphi_0(\zeta) = 0$ when $|\zeta| \to \infty$ and the integration is carried out subject to the condition $\lim \Phi_*(t) = 0$ when $|t| \to \infty$. On integrating in (23) along the upper and lower edges of the gaps, respectively, we obtain

$$\Phi^{\pm}(t) = -\frac{2}{\pi|t-\bar{t}|}\int_{\bar{t}}^{t}\varphi_0^{\pm}(\zeta)\sqrt{\frac{\zeta-\bar{t}}{\zeta-t}}d\zeta = S(\varphi_0^{\pm}(\zeta))$$

As before, we have

$$\Psi^{\pm}(t) = S(\psi_0^{\pm}(\zeta))$$

or, as follows from (13)

$$-\frac{2}{\pi|t-\bar{t}|}\int_{\bar{t}}^{t}(\gamma\varphi_0^{+}(\zeta)+\psi_0^{-}(\zeta))\sqrt{\frac{\zeta-\bar{t}}{\zeta-t}}d\xi = F_*(t) \quad (t \in L) \tag{24}$$

On applying the operator S^{-1}, which is the inverse of S [1] on the line $z = 0$ to both sides of (24), we obtain a matching problem for the holomorphic function $\varphi_0(\zeta)$

$$\gamma\varphi_0^{+}(\zeta)+\varphi_0^{-}(\zeta) = S^{-1}(F_*(t)) = f(\zeta) \quad (\zeta \in L) \tag{25}$$

Under the assumption that $f(\zeta)$ satisfies the Hölder condition when $|\zeta| \leqslant b_0$ and that $\lim \zeta\varphi_0(\zeta) = \lim \zeta\psi_0(\zeta) = 0$ when $|\zeta| \to \infty$, we write the solution of problem (25) as follows [5]

$$\varphi_0(\zeta) = \frac{(1+\beta)X(\zeta)}{2\pi i}\int_L \frac{f(\sigma)d\sigma}{X(\sigma)(\sigma-\zeta)} + X(\zeta)P_1(\zeta) \tag{26}$$

where by $X(t)$ we mean the values taken by the function

$$X(t) = (t-b_0)^{-1/2-i\lambda}(t+b_0)^{-1/2+i\lambda}$$

on the upper boundary L_+, $\lambda = -\ln(1+\beta)/(2\pi)$ and $P_1(\zeta) = C_1\zeta + C_2$, C_1, C_2 are unknown constants.

Since $\lim \zeta\varphi_0(\zeta) = 0$ when $|\zeta| \to \infty$ we have $C_1 = C_2 = 0$.

Now, on applying the operator S to both sides of (26) we obtain

$$\Phi_*(t) = S\left(\frac{(1+\beta)X(\zeta)}{2\pi i}\int_L \frac{f(\sigma)d\sigma}{X(\sigma)(\sigma-\zeta)}\right) = -\frac{2}{\pi|t-\bar{t}|}\int_L \varphi_0(\zeta)\sqrt{\frac{\zeta-\bar{t}}{\zeta-t}}d\zeta. \tag{27}$$

The function $\Psi_*(t)$ is found from (13) if relation (27) is used. The functions $\Phi_{**}(t)$ and $\Psi_{**}(t)$ are found in a similar way.

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